A Survey of Equal Sums of Like Powers

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Introduction. The Diophantine equation

$$(1) \qquad x_1^k + x_2^k + \cdots + x_m^k = y_1^k + y_2^k + \cdots + y_n^k, \qquad 1 \le m \le n,$$

has been studied by numerous mathematicians for many years and by various methods [1], [2]. We recently conducted a series of computer searches using the CDC 6600 to identify the sets of parameters k, m, n for which solutions exist and to find the least solutions for certain sets. This paper outlines the results of the computation, notes some previously published results, and concludes with a table showing, for various values of k and m, the least n for which a solution to (1) is known.

We restrict our attention to $k \le 10$. We assume that the x_i and y_j are positive integers and $x_i \ne y_j$. We do not distinguish between solutions which differ only in that the x_i or y_j are rearranged. We will refer to (1) as $(k.\ m.\ n)$ and say that a primitive solution to $(k.\ m.\ n)$ is one in which no integer > 1 divides all the numbers $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n$. Putting

$$z = \sum_1^m x_i^k = \sum_1^n y_j^k,$$

we order the primitive solutions according to the magnitude of z and denote the rth primitive solution to $(k.\ m.\ n)$ by $(k.\ m.\ n)_r$. Where we refer to the *range* covered in a search for solutions, we mean the upper limit on z. The notation $(x_1, x_2, \cdots, x_m)^k = (y_1, y_2, \cdots, y_n)^k$ means $\sum_1^m x_i^k = \sum_1^n y_j^k$. Any parametric solution discussed does not include all solutions unless otherwise stated.

Squares and Cubes. For $k = 2$ the general solution of the Pythagorean equation (2. 1. 2) is well known [3]. Many solutions in small integers and various parametric solutions have been given for (2. 1. n) with $n \ge 3$. The general solution of (2. 2. 2) is known [4]. Solutions to (2. 2. n) with $n \ge 3$ and (2. m. n) with $m \ge 3$ are numerous.

The impossibility of solving (k. 1. 2) with $k \ge 3$ is Fermat's last theorem, which has been established for $k \le 25000$ [5]. The general solution of (3. 1. 3) in rationals is attributed to Euler and Vieta [6] and also produces all solutions to (3. 2. 2) if the arguments are properly chosen. There are many solutions in small integers and various parametric solutions to (3. 1. n) with $n \ge 4$ and to (3. m. n) with $m \ge 2$ [7].

Fourth Powers.

(4. 1. n)—For $n = 3$, no solution is known. M. Ward [8] developed congruential constraints which, together with some hand computing, allowed him to show that $x^4 = y_1^4 + y_2^4 + y_3^4$ has no solution if $x \le 10{,}000$. The authors extended the search on the computer using a similar method and verified that there is no solution for $x \le 220{,}000$. Ward showed that if $x^4 = y_1^4 + y_2^4 + y_3^4$ is a primitive solution, it may be assumed that $x, y_1 \equiv 1 \pmod 2$, $y_2, y_3 \equiv 0 \pmod 8$ and either $x - y_1$ or $x + y_1$

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is $\equiv 0 \pmod{1024}$. Also $x \not\equiv 0 \pmod 5$ or else all y_i would be $\equiv 0 \pmod 5$ since $u^4 \equiv 0$ or 1 according as $u \equiv 0$ or $u \not\equiv 0 \pmod 5$. The computer program generated all numbers $M = (x^4 - y_1^4)/2048$ with $0 < y_1 < x$, x prime to 10 and $y_1 \equiv \pm x \pmod{1024}$. Tests were applied to $M = (y_2/8)^4 + (y_3/8)^4$ to reject cases in which a solution would not be primitive or M could not be the sum of two biquadrates. If M passed all the tests, its decomposition was attempted by trial using addition of entries in a stored table of biquadrates (27500 entries for $x \leq 220{,}000 = 8 \cdot 27500$). The tests were:

(1) M must be $\equiv 0$, 1 or 2 (mod 16) and (mod 5);

(2) M must not be $\equiv 7$, 8 or 11 (mod 13) and must not be $\equiv 4$, 5, 6, 9, 13, 22 or 28 (mod 29);

(3) x and y_1 must not both be divisible by an odd prime $p \equiv 3$, 5 or 7 (mod 8) for if so, p^4 divides M, p divides y_2 and y_3 and the solution is not primitive;

(4) M must not have a factor p where p is an odd prime not $\equiv 1 \pmod 8$ unless p^4 also divides M. In this case p divides y^2 and y^3, and in the decomposition by trial M can be replaced by M/p^4 (here tests were made only for $p < 100$).

Of approximately 19,200,000 initial values of M, only 22,400 required the trial decomposition.

TABLE I

Primitive solutions of (4. 1. 4) *for* $z \leq (8002)^4$

$$z = x_1^4 = \sum_1^4 y_j^4$$

i	x_1	y_1	y_2	y_3	y_4	Ref.
1	353	30	120	272	315	[9]
2	651	240	340	430	599	[34]
3	2487	435	710	1384	2420	[10]
4	2501	1130	1190	1432	2365	[10]
5	2829	850	1010	1546	2745	[10]
6	3723	2270	2345	2460	3152	[10]
7	3973	350	1652	3230	3395	[10]
8	4267	205	1060	2650	4094	[10]
9	4333	1394	1750	3545	3670	
10	4449	699	700	2840	4250	
11	4949	380	1660	1880	4907	
12	5281	1000	1120	3233	5080	
13	5463	410	1412	3910	5055	
14	5491	955	1770	2634	5400	[11]
15	5543	30	1680	3043	5400	
16	5729	1354	1810	4355	5150	
17	6167	542	2770	4280	5695	
18	6609	50	885	5000	5984	
19	6801	1490	3468	4790	6185	
20	7101	1390	2850	5365	6368	
21	7209	160	1345	2790	7166	
22	7339	800	3052	5440	6635	
23	7703	2230	3196	5620	6995	

For $n = 4$, R. Norrie [9] found the smallest solution $(353)^4 = (30, 120, 272, 315)^4$. J. O. Patterson [34] found $(4.1.4)_2$ and J. Leech [10] found the next 6 primitive solutions on the EDSAC 2 computer. S. Brudno [11] gave another primitive solution, the 14th in our Table I. The authors exhaustively searched the range 8002^4 using Leech's method finding in all the 23 primitives listed in Table I. No parametric solution has been found for (4. 1. 4) although the general solution is known for (3. 1. 3) and a parametric solution (discussed later) is known for (5. 1. 5).

TABLE II

Primitive solutions of (4. 2. 2) *for* $7.5 \times 10^{15} \leq z \leq 5.3 \times 10^{16}$

$$z = x_1^4 + x_2^4 = y_1^4 + y_2^4$$

i	x_1	x_2	y_1	y_2	z
*32	6262	8961	7234	8511	7 98564 45223 00177
33	5452	9733	7528	9029	9 85755 13638 85937
34	3401	10142	7054	9527	10 71400 42234 80497
35	5277	10409	8103	9517	12 51457 36160 92402
36	3779	10652	8332	9533	13 07827 22453 98097
37	3644	11515	5960	11333	17 75781 85225 58321
38	1525	12234	3550	12213	22 40674 37332 52161
**39	2903	12231	10203	10381	22 45039 16406 17602
40	1149	12653	7809	12167	25 63324 34950 11682
41	5121	13472	9153	12772	33 62808 84147 85537
42	5526	13751	11022	12169	36 68751 70593 08977
43	6470	14421	8171	14190	45 00187 64129 98081
44	6496	14643	11379	13268	47 75551 49900 03857
45	261	14861	8427	14461	48 77442 72266 31682
46	581	15109	8461	14723	52 11273 11403 26882

* For solutions to (4.2.2) for $i = 1$ to 31 see Lander and Parkin [18].

** This solution was found by Euler [37].

For $n \geq 5$ there exist many solutions in small integers. $(4.1.5)_1$ is $(5)^4 = (2, 2, 3, 4, 4)^4$. Several parametric solutions to (4. 1. 5) are known due to E. Fauquembergue [12], C. Haldeman [13], and A. Martin [14].

(4. 2. n)—For $n = 2$ the least solution is $(59, 158)^4 = (133, 134)^4$. Euler [15] gave a two-parameter solution and A. Gérardin [16] gave an equivalent but simpler form of this solution. Several of the smaller primitive solutions were found by Euler, A.

Werebrusow, and Leech [17] and a recent computer search by Lander and Parkin [18] extended the list of known primitives to 31. More recently we have increased this to a total of 46 primitives by a complete search of the range 5.3×10^{16} and the 15 new primitives are listed in Table II. The general solution is not known.

For $n \geq 3$ there are many small solutions. $(4.\ 2.\ 3)_1$ is $(7, 7)^4 = (3, 5, 8)^4$. Several parametric solutions are known for (4. 2. 3) due to Gérardin [19] and F. Ferrari [20].

$(4.\ m.\ n)$—For $m \geq 3$, solutions in small integers are numerous. Parametric solutions to (4. 3. 3) were given by Gérardin [21] and Werebrusow [22]. $(4.\ 3.\ 3)_1$ is $(2, 4, 7)^4 = (3, 6, 6)^4$.

Fifth Powers.

$(5.\ 1.\ n)$—For $n = 3$, no solution is known. Lander and Parkin [23], [24] found $(5.\ 1.\ 4)_1$ to be $(144)^5 = (27, 84, 110, 133)^5$. This disproved Euler's conjecture [25] that $(k.\ 1.\ n)$ has no solution if $1 < n < k$. No further primitive solutions to (5. 1. 4) exist in the range up to 765^5.

For $n = 5$, S. Sastry and S. Chowla [26] obtained a two-parameter solution yielding $(107)^5 = (7, 43, 57, 80, 100)^5$ as its minimal primitive; this solution is $(5.\ 1.\ 5)_3$. Lander and Parkin [24] found $(5.\ 1.\ 5)_1$ and $(5.\ 1.\ 5)_2$ to be $(72)^5 = (19, 43, 46, 47, 67)^5$ and $(94)^5 = (21, 23, 37, 79, 84)^5$. More recently we searched the range up to 599^5 and found in all the twelve primitive solutions given in Table III.

TABLE III

Primitive solutions of (5. 1. 5) *for* $z \leq 599^5$

$z = x_1^5 = \sum_1^5 y_j^5$

i	x_1	y_1	y_2	y_3	y_4	y_5	Ref.
1	72	19	43	46	47	67	[24]
2	94	21	23	37	79	84	[24]
3	107	7	43	57	80	100	[26]
4	365	78	120	191	259	347	
5	415	79	202	258	261	395	
6	427	4	26	139	296	412	
7	435	31	105	139	314	416	
8	480	54	91	101	404	430	
9	503	19	201	347	388	448	
10	530	159	172	200	356	513	
11	553	218	276	385	409	495	
12	575	2	298	351	474	500	

For $n \geq 6$ there are solutions in moderately small integers. $(5.\ 1.\ 6)_1$ is $(12)^5 = (4, 5, 6, 7, 9, 11)^5$ found by A. Martin [27]. The first eight primitive solutions to (5. 1. 6) are given in [24]. $(5.\ 1.\ 7)_1$ is $(23)^5 = (1, 7, 8, 14, 15, 18, 20)^5$.

(5. 2. n)—No solution is known for $n \leq 3$. An exhaustive search by the authors verified that there is no solution to (5. 2. 2) in the range up to 2.8×10^{14} or to (5. 2. 3) in the range up to 8×10^{12}. Sastry's parametric solution for (5. 1. 5) mentioned above gives for certain values of its arguments solutions to (5. 2. 4), the smallest being $(12, 38)^5 = (5, 13, 25, 37)^5$ which is $(5.\ 2.\ 4)_2$. K. Subba Rao [28] found $(3, 29)^5 = (4, 10, 20, 28)^5$ which is $(5.\ 2.\ 4)_1$. Table IV lists the ten primitives which exist in the range up to 2×10^{10}.

TABLE IV

Primitive solutions of (5. 2. 4) *for* $z \leq 2 \times 10^{10}$

$$z = \sum_1^2 x_j^5 = \sum_1^4 y_j^5$$

i	x_1	x_2	y_1	y_2	y_3	y_4	z	Ref.
1	3	29	4	10	20	28	205 11392	[28]
2	12	38	5	13	25	37	794 84000	[26]
3	28	52	26	29	35	50	3974 14400	
4	61	64	5	25	62	63	19183 38125	
5	16	85	6	50	53	82	44381 01701	
6	31	96	56	63	72	86	81823 56127	
7	14	99	44	58	67	94	95104 38323	
8	63	97	11	13	37	99	95797 76800	
9	25	106	48	57	76	100	1 33920 21401	
10	54	111	58	76	79	102	1 73097 46575	

For $n \geq 5$ there are solutions in moderately small integers; $(5.\ 2.\ 5)_1$ is $(1, 22)^5 = (4, 5, 7, 16, 21)^5$ due to Subba Rao [28]. We give the first six primitives for (5. 2. 5) in Table V.

(5. 3. n)—The first solution known for $n = 3$ was $(49, 75, 107)^5 = (39, 92, 100)^5$ due to A. Moessner [35]; this is $(5.\ 3.\ 3)_5$. H. P. F. Swinnerton-Dyer gave two separate two-parameter solutions [36]. We give the 45 primitives in the range up to 8×10^{12} in Table VI. For $n \geq 4$, solutions in small integers are plentiful. $(5.\ 3.\ 4)_1$ is $(3, 22, 25)^5 = (1, 8, 14, 27)^5$ due to Subba Rao [28]. A two-parameter solution to (5. 3. 4) was given by G. Xeroudakes and A. Moessner [29].

(5. m. n)—If $m \geq 4$, there are many solutions in small integers. $(5.\ 4.\ 4)_1$ is $(5, 6, 6, 8)^5 = (4, 7, 7, 7)^5$ due to Subba Rao [28]. Several parametric solutions to (5. 4. 4) were found by Xeroudakes and Moessner [29]. The first triple coincidence of four fifth powers is $1479604544 = (3, 48, 52, 61)^5 = (13, 36, 51, 64)^5 = (18, 36, 44, 66)^5$.

In the subsequent discussion we adopt a notation borrowed from the field of partitions, writing x^r to signify the term x repeated r times in the expression in which it appears. Table VII uses this notation, giving $(k.\ m.\ n)_1$ where known and references solutions in other tables. Table VII also shows for certain $(k.\ m.\ n)$ the range which has been searched on the computer exhaustively.

For the remainder of the equations $(k.\ m.\ n)$ which are discussed we note in the text only the limits searched, interesting features, and methods employed; specific solutions are given in Table VII.

Sixth Powers.

$(6.\ 1.\ n)$—No solution is known for $n \leq 6$. We consider the cases of $n = 6, 7$ and 8 in descending order. To solve $(6.\ 1.\ 8)$, $x^6 = \sum_1^8 y_i^6$, note that $u^6 \equiv 0$ or $1 \pmod 9$ according as $u \equiv 0$ or $u \not\equiv 0 \pmod 3$. Then if $x \equiv 0 \pmod 3$, all $y_i \equiv 0 \pmod 3$ and the solution is not primitive. Therefore take x and exactly one of the y_i (say y_1) prime to 3. Then $(x^6 - y_1^6)/3^6 = \sum_2^8 (y_i/3)^6$ is an integer (which is true if and only if $y_1 \equiv \pm x \pmod{243}$) to be decomposed by trial as the sum of 7 sixth powers. In Table VIII we give the 14 smallest primitives found by this method; $(6.\ 1.\ 8)_1$ is $(251)^6 = (8, 12, 30, 78, 102, 138, 165, 246)^6$.

TABLE V

Primitive solutions of (5. 2. 5) *for* $z \leq 2.8 \times 10^8$

$z = \sum_1^2 x_j^5 = \sum_1^5 y_j^5$

i	x_1	x_2	y_1	y_2	y_3	y_4	y_5	z
*1	1	22	4	5	7	16	21	51 53633
2	23	29	9	11	14	18	30	269 47492
3	16	38	10	14	26	31	33	802 83744
4	24	42	4	22	29	35	36	1386 53856
5	30	44	8	15	17	19	45	1892 16224
6	36	42	5	6	26	27	44	1911 57408

* The first solution is due to Subba Rao [28].

For $(6.\ 1.\ 7)$, $x^6 = \sum_1^7 y_i^6$, note that $u^6 \equiv 0$ or $1 \pmod 8$ according as u is even or odd. Then for a primitive solution, x and exactly one of the y_i are odd. The argument for $(6.\ 1.\ 8)$ modulo 9 applies and x is prime to 6, y_1 (say) is prime to 3, and either y_1 is odd or another y (say y_2), is odd. In the first case $y_1 \equiv \pm x \pmod{243}$ and (mod 32) and $(x^6 - y_1^6)/6^6 = \sum_2^7 (y_i/6)^6$ is an integer to be decomposed by trial as the sum of 6 sixth powers. In the second case $y_1 \equiv \pm x \pmod{243}$, $y_2 \equiv \pm x$ (mod 32) and $(x^6 - y_1^6 - y_2^6)/6^6 = \sum_3^7 (y_i/6)^6$ must be an integer (certain combinations x, y_1, y_2 satisfying the congruences are rejected) which is decomposed by trial as the sum of 5 sixth powers. The only solution for $x \leq 1536$ is $(6.\ 1.\ 7)_1$, $(1141)^6 = (74, 234, 402, 474, 702, 894, 1077)^6$ which is obtained in the second case.

TABLE VI

Primitive solutions of (5. 3. 3) *for* $z \leq 8 \times 10^{12}$

$$z = \sum_1^3 x_j^5 = \sum_1^3 y_j^5$$

i	x_1	x_2	x_3	y_1	y_2	y_3	z
1	24	28	67	3	54	62	13752 98099
2	18	44	66	13	51	64	14191 38368
3	21	43	74	8	62	68	23700 99168
4	56	67	83	53	72	81	58398 97526
*5	49	75	107	39	92	100	1 66810 39431
6	26	85	118	53	90	116	2 73265 12069
7	38	47	123	1	89	118	2 84616 37018
8	73	96	119	68	106	114	3 40903 35168
9	39	56	136	3	97	131	4 71668 30151
10	13	35	142	17	95	138	5 77882 32400
11	28	32	155	91	94	150	8 95168 61675
12	65	94	152	42	129	140	8 96361 42881
13	63	67	169	9	131	159	14 02010 53499
14	68	137	170	36	140	169	19 17013 58025
15	43	109	181	13	159	161	20 97974 92893
16	74	113	182	61	129	179	22 03336 44849
17	39	142	186	28	167	172	28 04458 41607
18	44	55	201	18	152	190	32 87486 01600
19	58	101	204	113	145	195	36 44723 14293
20	18	31	215	10	183	191	45 94319 03094
21	19	168	216	11	183	209	60 40152 82243
22	5	145	224	153	157	214	62 80466 82374
23	27	106	229	12	122	228	64 31599 96832
24	151	166	233	126	208	216	89 12718 82720
25	59	139	248	23	184	239	99 07237 88966
26	157	193	234	147	218	219	106 47575 48174
27	2	97	258	35	125	257	115 17249 93057
28	3	121	264	163	185	250	130 83259 82668
29	97	181	274	67	227	258	174 72267 67782
30	99	105	286	30	179	281	193 57802 02300
31	132	154	283	80	219	270	194 19238 97099
32	106	137	288	201	219	261	204 29996 35401
33	40	168	289	3	215	279	214 99241 22017
34	136	158	294	71	249	268	234 15192 15168
35	193	229	282	179	259	266	268 09353 50774
36	107	229	293	93	259	277	280 32137 94149
37	31	173	307	7	201	303	288 20348 39551
38	102	118	310	49	270	271	289 68334 85600
39	116	124	310	21	235	294	291 32347 67200
40	30	39	331	65	224	321	397 33103 34850
41	119	232	328	89	289	301	449 23488 61399
42	108	181	348	53	246	338	531 27877 53637
43	114	211	364	52	298	339	682 75705 13699
44	172	206	364	102	303	337	691 15935 15232
45	123	137	373	13	259	361	729 65305 14393

* This solution was found by A. Moessner [35].

TABLE VII

(k. m. n)₁ and summary of results

(k. m. n)	Range Searched	Solutions Known*
4. 1. 3	2.34×10^{21}	None known
4. 1. 4	4.1×10^{15}	$(353)^4 = (30, 120, 272, 315)^4$
		See Table I, 23 solutions
4. 1. 5		$(5)^4 = (2^2, 3, 4^2)^4$
4. 2. 2	5.3×10^{16}	$(59, 158)^4 = (133, 134)^4$
		See Table I in [18], and Table II, 46 solutions
4. 2. 3		$(7^2)^4 = (3, 5, 8)^4$
4. 3. 3		$(2, 4, 7)^4 = (3, 6^2)^4$
5. 1. 3	2.6×10^{14}	None known
5. 1. 4	2.6×10^{14}	$(144)^5 = (27, 84, 110, 133)^5$
5. 1. 5	7.7×10^{13}	$(72)^5 = (19, 43, 46, 47, 67)^5$
		See Table III, 12 solutions
5. 1. 6		$(12)^5 = (4, 5, 6, 7, 9, 11)^5$
5. 1. 7		$(23)^5 = (1, 7, 8, 14, 15, 18, 20)^5$
5. 2. 2	2.8×10^{14}	None known
5. 2. 3	8×10^{12}	None known
5. 2. 4	2×10^{10}	$(3, 29)^5 = (4, 10, 20, 28)^5$
		See Table IV, 10 solutions
5. 2. 5	2×10^{8}	$(1, 22)^5 = (4, 5, 7, 16, 21)^5$
		See Table V, 6 solutions
5. 3. 3	8×10^{12}	$(24, 28, 67)^5 = (3, 54, 62)^5$
		See Table VI, 45 solutions
5. 3. 4		$(3, 22, 25)^5 = (1, 8, 14, 27)^5$
5. 4. 4		$(5, 6^2, 8)^5 = (4, 7^3)^5$
6. 1. n	3.16×10^{27}	None known for $n \leq 6$
6. 1. 7	1.3×10^{19}	$(1141)^6 = (74, 234, 402, 474, 702, 894, 1077)^6$
6. 1. 8	5.8×10^{16}	$(251)^6 = (8, 12, 30, 78, 102, 138, 165, 246)^6$
		See Table VIII, 14 solutions
6. 1. 9		$(54)^6 = (1, 17, 19, 22, 31, 37^2, 41, 49)^6$
6. 1. 10		$(39)^6 = (2, 4, 7, 14, 16, 26^2, 30, 32^2)^6$
6. 1. 11		$(18)^6 = (2, 5^3, 7^2, 9^2, 10, 14, 17)^6$
6. 2. n	4×10^{12}	None known for $n \leq 6$
6. 2. 7		$(56, 91)^6 = (18, 22, 36, 58, 69, 78^2)^6$
6. 2. 8		$(35, 37)^6 = (8, 10, 12, 15, 24, 30, 33, 36)^6$
6. 2. 9		$(6, 21)^6 = (1, 5^2, 7, 13^3, 17, 19)^6$
6. 2. 10		$(12^2)^6 = (1^3, 4^2, 7, 9, 11^3)^6$
6. 3. 3	2.5×10^{14}	$(3, 19, 22)^6 = (10, 15, 23)^6$
		See Table IX, 10 solutions
6. 3. 4	2.9×10^{12}	$(41, 58, 73)^6 = (15, 32, 65, 70)^6$
		See Table X, 5 solutions
6. 4. 4		$(2^2, 9^2)^6 = (3, 5, 6, 10)^6$
7. 1. n	1.95×10^{14}	None known for $n \leq 7$
7. 1. 8		$(102)^7 = (12, 35, 53, 58, 64, 83, 85, 90)^7$
7. 1. 9		$(62)^7 = (6, 14, 20, 22, 27, 33, 41, 50, 59)^7$
7. 2. 8		$(10, 33)^7 = (5, 6, 7, 15^2, 20, 28, 31)^7$
7. 3. 7		$(26, 30^2)^7 = (7^2, 12, 16, 27, 28, 31)^7$
7. 4. 5		$(12, 16, 43, 50)^7 = (3, 11, 26, 29, 52)^7$
7. 5. 5		$(8^2, 13, 16, 19)^7 = (2, 12, 15, 17, 18)^7$
		See Table XI, 17 solutions

* All solutions shown are $(k.\ m.\ n)_1$ unless otherwise marked.

TABLE VII (*cont.*)

(k. m. n)	Range Searched	Solutions Known
7. 6. 6		$(2, 3, 6^2, 10, 13)^7 = (1^2, 7^2, 12^2)^7$
8. 1. 11		$(125)^8 = (14, 18, 44^2, 66, 70, 92, 93, 96, 106, 112)^8$
8. 1. 12		$(65)^8 = (8^2, 10, 24^3, 26, 30, 34, 44, 52, 63)^8$
8. 2. 9		$(11, 27)^8 = (2, 7, 8, 16, 17, 20^2, 24^2)^8$
8. 3. 8		$(8, 17, 50)^8 = (6, 12, 16^2, 38^2, 40, 47)^8$
8. 4. 7		$(6, 11, 20, 35)^8 = (7, 9, 16, 22^2, 28, 34)^8$
8. 5. 5		$(1, 10, 11, 20, 43)^8 = (5, 28, 32, 35, 41)^8$
8. 6. 6		$(3, 6, 8, 10, 15, 23)^8 = (5, 9^2, 12, 20, 22)^8$
8. 7. 7		$(1, 3, 5, 6^2, 8, 13)^8 = (4, 7, 9^2, 10, 11, 12)^8$
8. 8. 8		$(1, 3, 7^3, 10^2, 12)^8 = (4, 5^2, 6^2, 11^3)^8$
9. 1. 15		$(26)^9 = (2^2, 4, 6^2, 7, 9^2, 10, 15, 18, 21^2, 23^2)^9$
9. 2. 12		$(15, 21)^9 = (2^4, 3^2, 4, 7, 16, 17, 19^2)^9$
9. 3. 11		$(13, 16, 30)^9 = (2, 3, 6, 7, 9^2, 19^2, 21, 25, 29)^9$
9. 4. 10		$(5, 12, 16, 21)^9 = (2, 6^2, 9, 10, 11, 14, 18, 19^2)^9$
9. 5. 11		$(7, 8, 14, 20, 22)^9 = (3, 5^2, 9^2, 12, 15^2, 16, 21^2)^9$
9. 6. 6		$(1, 13^2, 14, 18, 23)^9 = (5, 9, 10, 15, 21, 22)^9$
10. 1. 23		$(15)^{10} = (1^5, 2, 3, 6, 7^6, 9^4, 10, 12^2, 13, 14)^{10}$
10. 2. 19		$(9, 17)^{10} = (2^5, 5, 6, 10, 11^6, 12^2, 15^3)^{10}$
10. 3. 24		$(11, 15^2)^{10} = (1, 2, 3, 4^{10}, 7, 8^7, 10, 12, 16)^{10}$
10. 4. 23		$(11^3, 16)^{10} = (1^5, 2^2, 3^2, 4, 6^4, 7^3, 8, 10^2, 14^2, 15)^{10}$
10. 5. 16		$(3^2, 8, 14, 16)^{10} = (1^4, 2, 4^2, 6, 12^2, 13^5, 15)^{10}$
10. 6. 27		$(2^2, 8, 11, 12^2)^{10} = (1, 3^4, 4^2, 5^2, 6^7, 7^9, 10, 13)^{10}$
*10. 7. 7		$(1, 28, 31, 32, 55, 61, 68)^{10} = (17, 20, 23, 44, 49, 64, 67)^{10}$

* Moessner [35]; not known to be $(10.\ 7.\ 7)_1$.

For (6. 1. 6), $x^6 = \sum_1^6 y_i^6$ note that $u^6 \equiv 0$ or 1 (mod 7) according as $u \equiv 0$ or $u \not\equiv 0$ (mod 7). Then for a primitive solution, x and exactly one of the y_i (say y_1) are prime to 7. This implies $y_1 \equiv \pm x$, $\pm qx$ or $\pm q^2x$ where $q = 34968$ is a primitive sixth root of unity (mod $7^6 = 117649$). Now the foregoing arguments modulo 8 and modulo 9 apply, and there are five cases.

(1) If $y_1 \equiv \pm 1$ (mod 6) then $y_1 \equiv \pm x$ (mod 243) and (mod 32) and $(x^6 - y_1^6)/42^6 = \sum_2^6 (y_i/42)^6$ is an integer to be decomposed by trial as the sum of 5 sixth powers.

(2) If $y_1 \equiv \pm 2$ (mod 6) then $y_1 \equiv \pm x$ (mod 243) and another of the y_i (say y_2), is odd. Then $y_2 \equiv 0$ (mod $3 \cdot 7$), $y_2 \equiv \pm x$ (mod 32), and $(x^6 - y_1^6 - y_2^6)/42^6 = \sum_3^6 (y_i/42)^6$ is the sum of 4 integral sixth powers.

(3) If $y_1 \equiv 3$ (mod 6) then $y_1 \equiv \pm x$ (mod 32) and another of the y_i (say y_2), is prime to 3, $y_2 \equiv 0$ (mod $2 \cdot 7$), and $y_2 \equiv \pm x$ (mod 243). In case (2), $(x^6 - y_1^6 - y_2^6)/42^6$ is an integer and is the sum of 4 sixth powers.

(4) If $y_1 \equiv 0$ (mod 6), another of the y_i (say y_2), is prime to 3, $y_2 \equiv 0$ (mod 7) and $y_2 \equiv \pm x$ (mod 243). If y_2 is odd, then $y_2 \equiv \pm x$ (mod 32) and as in cases (2) and (3) $(x^6 - y_1^6 - y_2^6)/42^6$ is the sum of 4 sixth powers. If y_2 is even, we have case (5).

(5) Another of the y_i (say y_3), is odd, $y_3 \equiv 0$ (mod $3 \cdot 7$), $y_3 \equiv \pm x$ (mod 32), and $(x^6 - y_1^6 - y_2^6 - y_3^6)/42^6 = \sum_4^6 (y_i/42)^6$ is an integer to be decomposed as the sum of 3 sixth powers.

The search for a solution to (6. 1. 6) was carried exhaustively by this method through the range $x \leq 38314$ and there is no solution in this range.

A. Martin [30] gave a solution to (6. 1. 16); Moessner [31] gave solutions to (6. 1. n) for $n = 16, 18, 20$ and 23. For $n \geq 11$, it is not difficult to find solutions in small integers.

TABLE VIII

Primitive Solutions of (6. 1. 8) *for* $z \leq 7 \times 10^{16}$

$$z = x_1^6 = \sum_1^8 y_i^6$$

i	x_1	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
1	251	8	12	30	78	102	138	165	246
2	431	48	111	156	186	188	228	240	426
3	440	93	93	195	197	303	303	303	411
4	440	219	255	261	267	289	351	351	351
5	455	12	66	138	174	212	288	306	441
6	493	12	48	222	236	333	384	390	426
7	499	66	78	144	228	256	288	435	444
8	502	16	24	60	156	204	276	330	492
9	547	61	96	156	228	276	318	354	534
10	559	170	177	276	312	312	408	450	498
11	581	60	102	126	261	270	338	354	570
12	583	57	146	150	360	390	402	444	528
13	607	33	72	122	192	204	390	534	534
14	623	12	90	114	114	273	306	492	592

(6. 3. n)—Subba Rao [32] found the solution $(3, 19, 22)^6 = (10, 15, 23)^6$ which is $(6.\ 3.\ 3)_1$. In Table IX we give the remaining 9 primitive solutions which exist in the range up to 2.5×10^{14}. It is interesting to note that each of the solutions except the sixth is also a solution to (2. 3. 3). Table X gives the five primitive solutions to (6. 3. 4) which exist in the range up to 2.9×10^{12}.

(6. m. n)—If m is ≥ 4, solutions in small integers can be found readily. Subba Rao [32] gave $(6.\ 4.\ 4)_1$ (see Table VII). The first triple coincidence of 4 sixth powers is $1885800643779 = (1, 34, 49, 111)^6 = (7, 43, 69, 110)^6 = (18, 25, 77, 109)^6$.

Seventh Powers.

$(7.\ 2.\ 10)_2$ is $(2, 27)^7 = (4, 8, 13, 14^2, 16, 18, 22, 23^2)^7 = (7^2, 9, 13, 14, 18, 20, 22^2, 23)^7$ which is a double primitive and reduces to the solution $(7.\ 5.\ 5)_2$.

TABLE IX

Primitive solutions of (6. 3. 3) *for* $z \leq 2.5 \times 10^{14}$

$$z = \sum_1^3 x_j^6 = \sum_1 y_j^6$$

i	x_1	x_2	x_3	y_1	y_2	y_3	z
*1	3	19	22	10	15	23	1604 26514
2	36	37	67	15	52	65	9 52008 90914
3	33	47	74	23	54	73	17 62771 73474
4	32	43	81	3	55	80	28 98246 41354
5	37	50	81	11	65	78	30 06202 62890
6	25	62	138	82	92	135	696 38068 13393
7	51	113	136	40	125	129	842 70669 28346
8	71	92	147	1	132	133	1082 47536 54794
9	111	121	230	26	169	225	15304 47319 28882
10	75	142	245	14	163	243	22464 65092 02194

* The first solution is due to K. Subba Rao [32].

TABLE X

Primitive solutions of (6. 3. 4) *for* $z \leq 2.9 \times 10^{12}$

$$z = \sum_1^3 x_j^6 = \sum_1^4 y_j^6$$

i	x_1	x_2	x_3	y_1	y_2	y_3	y_4	z
1	41	58	73	15	32	65	70	19 41530 23074
2	61	62	85	52	56	69	83	48 54701 25570
3	61	74	85	26	56	71	87	59 28763 80162
4	11	88	90	21	74	78	92	99 58468 58345
5	26	83	95	23	24	28	101	106 23411 79770

(7. 5. n)—Table XI lists the 17 primitive solutions to (7. 5. 5) which exist in the range up to 4.0×10^{12}.

Eighth Powers.

(8. 1. n)—We found a parametric solution to (8. 1. 17), $(2^{8k+4} + 1)^8 = (2^{8k+4} - 1)^8 + (2^{7k+4})^8 + (2^{k+1})^8 + 7[(2^{5k+3})^8 + (2^{3k+2})^8]$ which for $k = 0$ yields $(8.\ 1.\ 17)_1$. This was the solution used by Sastry [26] in developing a parametric solution to (8. 8. 8). The computer program used in searching for solutions to (8. 1. n) was based on the congruences $x^8 \equiv 0$ or 1 (mod 32) according as $x \equiv 0$ or 1 (mod 2) so that primitive solutions to $x^8 = \sum_1^n y_j^8$ with $n < 32$ must have x and (say) y_1 both odd. Then $x^8 - y_1^8$ is divisible by 2^8 which implies $x \equiv \pm y_1$ (mod 32), and $(x^8 - y_1^8)/256$ is decomposed as the sum of $n - 1$ eighth powers by trial.

Solutions to (8. 5. 5) and (8. 9. 9) were found by A. Letac [33].

Ninth and Tenth Powers. Computations performed by the authors for (9.*m*.*n*) and (10. *m*. *n*) are the basis for the data shown in the last two columns of Table XII,

TABLE XI

Primitive solutions of (7. 5. 5) *for* $z \leq 4.0 \times 10^{12}$

$$z = \sum_1^5 x_j^7 = \sum_1^5 y_j^7$$

i	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	z
1	8	8	13	16	19	2	12	15	17	18	12292 50016
2	4	8	14	16	23	7	7	9	20	22	37807 87943
3	11	12	18	21	26	9	10	22	23	24	1 05004 37728
4	6	12	20	22	27	10	13	13	25	26	1 42708 22835
5	3	13	17	24	38	14	26	32	32	33	11 94751 43393
6	4	5	30	36	44	2	8	27	39	43	41 95120 68269
7	16	33	33	33	44	18	26	34	38	43	44 74015 74051
8	3	4	21	39	45	14	23	33	41	43	51 27015 66916
9	16	17	26	33	49	10	12	30	43	46	72 95521 00131
10	15	18	18	43	48	8	11	32	44	47	86 02822 52818
11	19	24	43	46	51	9	36	40	48	50	161 05272 89337
12	13	16	35	35	56	9	19	28	44	55	185 61046 27259
13	9	11	43	45	55	3	19	37	51	53	216 79475 68747
14	9	15	19	34	59	5	10	16	48	57	254 22443 49046
15	23	27	40	49	56	7	39	45	51	53	258 30231 01035
16	8	13	41	45	59	2	10	47	52	55	305 71400 57494
17	1	38	39	39	60	8	25	34	53	57	318 82375 95951

TABLE XII

Least n for which a solution to (*k*. *m*. *n*) *is known*

	k								
m	2	3	4	5	6	7	8	9	10
1	2	3	4	4	7	8	11	15	23
2	2	2	2	4	7	8	9	12	19
3				3	3	7	8	11	24
4						5	7	10	23
5						5	5	11	16
6								6	27
7									7

except for a solution to (10. 7. 7) given by A. Moessner [35]. Due to computer word length limitations the calculations were not extended to large values of the arguments.

Additional References. A. Gloden gave a parametric solution of (5. 4. 4) in [38], two parametric solutions of (7. 5. 5) in [39], [40], and a parametric solution of (8. 7. 7) in [41]. A. Moessner gave numerical solutions of (5. 2. 4) and (5. 3. 3) in [42]. In [43] Moessner gave three parametric solutions of (6. 4. 4) and parametric solutions of (8. 7. 7) and (9. 10. 10). Two numerical solutions of (7. 4. 5) due to A. Letac are found in [39]. S. Sastry and T. Rai solved (7. 6. 6) parametrically [44].

G. Palamà [45] gave numerical solutions of (9. 11. 11) and (11. 10. 12). In [46] Moessner and Gloden solved (8. 6. 6) and (8. 6. 7) numerically.

Concluding Remarks. Let $N(k,m)$ be the smallest n for which $(k.m.n)$ is solvable. In Table XII we show the upper bound to N based on the results just presented. Each column is terminated when a solution to $(k.\ m.\ m)$ has been found. It appears likely that whenever $(k.\ m.\ m)$ is solvable, so is $(k.\ r.\ r)$ for any $r > m$. Some questions are:

(a) Is $N(k, m + 1) \leq N(k, m) \leq N(k + 1, m)$ always true?

(b) Is $(k.\ m.\ n)$ always solvable when $m + n > k$?

(c) Is it true that $(k.\ m.\ n)$ is never solvable when $m + n < k$?

(d) For which k, m, n such that $m + n = k$ is $(k.\ m.\ n)$ solvable?

The results presented in this paper tend to support an affirmative answer to (c). Question (d) appears to be especially difficult. The only solvable cases with $m + n = k$ known at present are (4. 2. 2), (5. 1. 4) and (6. 3. 3).

In this paper we have made a computational attack on the problem of finding a sum of n kth powers which is also the sum of a smaller number of kth powers. In many of the cases considered, especially for the larger values of k, we have undoubtedly not obtained the best possible results, but the amount of computing needed to do this would seem to be overwhelming.

We believe that the main result of this paper is the presentation of results on a family of Diophantine equations which have largely been considered separately in the past. We hope that this presentation offers greater insight into the nature of the function $N(k, m)$ and that future efforts will be directed toward reducing the upper bounds for this function.

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